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# First-Order Impulsive Partial Differential Inequalities

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We deal with the Cauchy problem for nonlinear impulsive partial differential equations of first order. Theorems on impulsive differential inequalities are obtained. Comparison results implying uniqueness criteria are proved.

## 1. INTRODUCTION

Most evolution processes in nature are characterized by the fact that at fixed moments of time the parameters of the system are abruptly changed. This was the reason for the development of the theory of impulsive ordinary differential equations, and this theory has been elaborated to a considerable extent (Bainov *et al.*, 1989; Bainov and Simeonov, 1989).

First-order partial differential equations have the following property: the problem of existence of their classical solutions is closely connected with the problem of solving systems of ordinary differential equations. Ordinary differential inequalities find numerous applications in the theory of first-order partial differential equations. Such problems as estimates of solutions of partial differential equations, estimates of the domain of the solution, estimates of the difference between two solutions, and criteria for uniqueness are classic examples (Kamont, 1979; Lakshmikantham and Leela, 1969; Ladde *et al.*, 1985; Szarski, 1965). We note that the development of the theory of partial differential inequalities is connected with the names of V. Lakshmikantham, S. Leela, and J. Szarski.

We start the investigation of the corresponding theory for first-order impulsive partial differential equations. We prove natural generalizations of

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theorems on differential inequalities. We show that impulsive ordinary differential inequalities find application in proofs of theorems concerning the estimate of solutions and in the uniqueness theory of impulsive partial differential equations. We note that the start of the theory of impulsive partial differential equations was made in Erbe *et al.* (1991), Rogovchenko (1988), and Rogovchenko and Trofimchuk (1986).

## 2. PRELIMINARY NOTES

We denote by $C(X, Y)$ the class of all continuous functions from $X$ into $Y$, where $X$ and $Y$ are metric spaces. We will be using vectorial inequalities, understanding that the same inequalities hold between their respective components. Suppose that $a > 0$,

$$\alpha = (\alpha_1, \ldots, \alpha_n)\colon \quad [0, a) \to R^n$$

$$\beta = (\beta_1, \ldots, \beta_n)\colon \quad [0, a) \to R^n$$

are given functions and $\alpha(x) < \beta(x)$ for $x \in [0, a)$.

Let

$$E = \{(x, y) \in R^{1+n}\colon x \in [0, a), \alpha(x) \leq y \leq \beta(x)\}$$

Suppose that

$$0 < x_1 < x_2 < \cdots < x_k < a$$

are given numbers and $x_0 = 0$, $x_{k+1} = a$.

We define

$$\Gamma_i = \{(x, y) \in E\colon x_i < x < x_{i+1}\}, \qquad i = 0, 1, \ldots, k$$

and $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_k$.

Let $C_{\text{imp}}[E, R]$ be the class of all functions $Z\colon E \to R$ such that:

(i) The functions $Z|_{\Gamma_i}$, $i = 0, 1, \ldots, k$, are continuous.

(ii) For each $i$, $1 \leq i \leq k$, $x = x_i$, there exists

$$\lim_{\substack{(t,s) \to (x,y) \\ t < x}} Z(t, s) = Z(x^-, y), \qquad \alpha(x) \leq y \leq \beta(x)$$

(iii) For each $i$, $0 \leq i \leq k$, $x = x_i$, there exists

$$\lim_{\substack{(t,s) \to (x,y) \\ t > x}} Z(t, s) = Z(x^+, y), \qquad \alpha(x) \leq y \leq \beta(x)$$

(iv) For each $i$, $0 \leq i \leq k$, $x = x_i$, we have

$$Z(x, y) = Z(x^+, y), \qquad \alpha(x) \leq y \leq \beta(x)$$

For a function $Z \in C_{\text{imp}}[E, R]$ we define

$$\Delta Z(x_i, y) = Z(x_i, y) - Z(x_i^-, y), \qquad i = 1, \ldots, k$$

Let $\Omega = E \times R \times R^n$, $\tilde{\Omega} = E \times R$. Suppose that $f: \Omega \to R$, $g = (g_1, \ldots, g_k): \tilde{\Omega} \to R^k$, $\phi: [\alpha(0), \beta(0)] \to R$ are given functions. We consider the Cauchy problem

$$Z_x(x, y) = f(x, y, Z(x, y), Z_y(x, y)) \tag{1}$$

$$Z(0, y) = \phi(y), \qquad y \in [\alpha(0), \beta(0)] \tag{2}$$

$$\Delta Z(x_i, y) = g_i(x_i, y, Z(x_i^-, y)), \qquad y \in [\alpha(x_i), \beta(x_i)] \tag{3}$$

$$i = 1, \ldots, k; \qquad z_y(x, y) = (z_{y_1}(x, y), \ldots, z_{y_n}(x, y))$$

*Definition 1.* A function $Z: E \to R$ is a *solution* of (1)–(3) if

(i) $Z \in C_{\text{imp}}[E, R]$, there exist derivatives $Z_x(x, y)$ and $Z_y(x, y)$ for $(x, y) \in \Gamma$, and $Z$ satisfies (1) on $\Gamma$.

(ii) $Z$ satisfies (2) and (3).

Let

$$S = \bigcup_{i=0}^{k} \{\partial\Gamma_i \cap [(x_i, x_{i+1}) \times R^n]\}$$

A function $Z \in C_{\text{imp}}[E, R]$ will be called a *function of class* $C^*_{\text{imp}}[E, R]$ if $Z$ has partial derivatives $Z_x(x, y)$ and $Z_y(x, y)$ for $(x, y) \in \Gamma$ and there exists the total derivative of $Z$ on $S$.

We define the functions $I_0$, $I_+$, and $I_-$ as follows. For each $(x, y) \in E$ there exist sets of integers $I_0[x, y]$, $I_+[x, y]$, and $I_-[x, y]$ such that

$$I_0[x, y] \cup I_+[x, y] \cup I_-[x, y] = \{1, \ldots, n\}$$

and

$$y_i = \alpha_i(x) \qquad \text{for} \quad i \in I_-[x, y]$$

$$y_i = \beta_i(x) \qquad \text{for} \quad i \in I_+[x, y]$$

$$\alpha_i(x) < y_i < \beta_i(x) \qquad \text{for} \quad i \in I_0[x, y]$$

## 3. MAIN RESULTS

### 3.1. Differential Inequalities with Impulses

In this part of the paper we consider differential inequalities generated by (1)–(3).

We introduce the following assumptions.

H1. $\alpha, \beta \in C([0, a), R^n)$ and the functions $\alpha|_{I_i}$ and $\beta|_{I_i}$, where $I_i = (x_i, x_{i+1})$, $i = 0, 1, \ldots, k$, are of class $C^1$.

H2. $\alpha(x) < \beta(x)$ for $x \in [0, a)$.

H3. $f: \Omega \to R$ and for $(x, y) \in S$, $p \in R$, we have

$$f(x, y, p, q) - f(x, y, p, \bar{q}) + \sum_{i \in I_-[x,y]} \alpha_i'(x)(q_i - \bar{q}_i) + \sum_{i \in I_+[x,y]} \beta_i'(x)(q_i - \bar{q}_i) \le 0 \tag{4}$$

where $q = (q_1, \ldots, q_n)$, $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$, and

$$q_i \le \bar{q}_i \quad \text{for} \quad i \in I_-[x, y]$$
$$q_i \ge \bar{q}_i \quad \text{for} \quad i \in I_+[x, y]$$
$$q_i = \bar{q}_i \quad \text{for} \quad i \in I_0[x, y]$$

H4. For each $i$, $1 \le i \le k$, the functions $g_i: \tilde{\Omega} \to R$ are such that $\delta_i(=p^+) + g_i(x, y, p)$ are nondecreasing on $R$.

*Theorem 1.* Suppose that the following conditions hold:

1. Assumptions H1–H4 are satisfied.
2. $U, V \in C^*_{\text{imp}}[E, R]$ satisfy

$$U(0, y) < V(0, y) \qquad \text{for} \quad \alpha(0) \le y \le \beta(0) \tag{5}$$

3. The differential inequality

$$U_x(x, y) - f(x, y, U(x, y), U_y(x, y)) < V_x(x, y) - f(x, y, V(x, y), V_y(x, y)) \tag{6}$$

holds true on $\Gamma$ and

$$\Delta U(x_i, y) - g_i(x_i, y, U(x_i^-, y)) < \Delta V(x_i, y) - g_i(x_i, y, V(x_i^-, y)) \tag{7}$$

where $\alpha(x_i) \le y \le \beta(x_i)$, $i = 1, \ldots, k$.

Then we have

$$U(x, y) < V(x, y) \qquad \text{for} \quad (x, y) \in E \tag{8}$$

*Proof.* If assertion (8) is false, then the set

$$Z = \{x \in [0, a): \text{there exists } y \in [\alpha(x), \beta(x)] \text{ such that } U(x, y) \ge V(x, y)\}$$

is nonempty. Defining $\tilde{x} = \inf Z$, it follows from (5) that $\tilde{x} > 0$ and there exists $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n) \in [\alpha(\tilde{x}), \beta(\tilde{x})]$ such that

$$\begin{aligned} &U(x, y) < V(x, y) \qquad \text{for} \quad (x, y) \in E \cap ([0, \tilde{x}) \times R^n) \\ &U(\tilde{x}, \tilde{y}) = V(\tilde{x}, \tilde{y}) \end{aligned} \tag{9}$$

Now, there are three cases to be distinguished.

*Case 1.* $(\tilde{x}, \tilde{y}) \in \operatorname{Int} \Gamma$. Then

$$U_y(\tilde{x}, \tilde{y}) = V_y(\tilde{x}, \tilde{y})$$
$$U_x(\tilde{x}, \tilde{y}) \geq V_x(\tilde{x}, \tilde{y})$$

which leads to a contradiction with (6).

*Case 2.* $(\tilde{x}, \tilde{y}) \in S$. Then we have

$$\begin{aligned} U_{y_i}(\tilde{x}, \tilde{y}) - V_{y_i}(\tilde{x}, \tilde{y}) &\geq 0 \quad \text{for} \quad i \in I_+[\tilde{x}, \tilde{y}] \\ U_{y_i}(\tilde{x}, \tilde{y}) - V_{y_i}(\tilde{x}, \tilde{y}) &\leq 0 \quad \text{for} \quad i \in I_-[\tilde{x}, \tilde{y}] \\ U_{y_i}(\tilde{x}, \tilde{y}) - V_{y_i}(\tilde{x}, \tilde{y}) &= 0 \quad \text{for} \quad i \in I_0[\tilde{x}, \tilde{y}] \end{aligned} \tag{10}$$

For $x \in [0, \tilde{x}]$ we put

$$r(x) = (r_1(x), \ldots, r_n(x))$$

where

$$\begin{aligned} r_i(x) &= \alpha_i(x) \quad \text{for} \quad i \in I_-[\tilde{x}, \tilde{y}] \\ r_i(x) &= \beta_i(x) \quad \text{for} \quad i \in I_+[\tilde{x}, \tilde{y}] \\ r_i(x) &= \tilde{y}_i \quad \text{for} \quad i \in I_0[\tilde{x}, \tilde{y}] \end{aligned} \tag{11}$$

We consider the composite function $s(x) = U(x, r(x)) - V(x, r(x))$, $x \in [0, \tilde{x}]$. It attains its maximum at $x = \tilde{x}$ and therefore

$$\begin{aligned} U_x(\tilde{x}, \tilde{y}) - V_x(\tilde{x}, \tilde{y}) &+ \sum_{i \in I_-[\tilde{x}, \tilde{y}]} \alpha_i'(\tilde{x})[U_{y_i}(\tilde{x}, \tilde{y}) - V_{y_i}(\tilde{x}, \tilde{y})] \\ &+ \sum_{i \in I_+[\tilde{x}, \tilde{y}]} \beta_i'(\tilde{x})[U_{y_i}(\tilde{x}, \tilde{y}) - V_{y_i}(\tilde{x}, \tilde{y})] \geq 0 \end{aligned} \tag{12}$$

From (4), (6), (9), and (10) it follows that

$$\begin{aligned} &U_x(\tilde{x}, \tilde{y}) - V_x(\tilde{x}, \tilde{y}) \\ &\quad < f(\tilde{x}, \tilde{y}, U(\tilde{x}, \tilde{y}), U_y(\tilde{x}, \tilde{y})) - f(\tilde{x}, \tilde{y}, V(\tilde{x}, \tilde{y}), V_y(\tilde{x}, \tilde{y})) \\ &\quad \leq -\sum_{i \in I_-[\tilde{x}, \tilde{y}]} \alpha_i'(\tilde{x})[U_{y_i}(\tilde{x}, \tilde{y}) - V_{y_i}(\tilde{x}, \tilde{y})] \\ &\quad - \sum_{i \in I_+[\tilde{x}, \tilde{y}]} \beta_i'(\tilde{x})[U_{y_i}(\tilde{x}, \tilde{y}) - V_{y_i}(\tilde{x}, \tilde{y})] \end{aligned}$$

which contradicts (12).

*Case 3.* $(\tilde{x}, \tilde{y}) \in E \backslash \Gamma$. Then there exists $i$, $1 \leq i \leq k$, such that $\tilde{x} = x_i$. Then we have from (9)

$$U(\tilde{x}^-, \tilde{y}) - V(\tilde{x}^-, \tilde{y}) \leq 0 \tag{13}$$

It follows from (7), (13) that

$$U(\tilde{x}, \tilde{y}) - V(\tilde{x}, \tilde{y}) < U(\tilde{x}^-, \tilde{y}) + g_i(\tilde{x}, \tilde{y}, U(\tilde{x}^-, \tilde{y})) \\ -[V(\tilde{x}^-, \tilde{y}) + g_i(\tilde{x}, \tilde{y}, V(\tilde{x}^-, \tilde{y}))] \leq 0$$

which contradicts (9).

Hence $Z$ is empty and statement (8) follows. ■

*Remark 1.* Analogous results for the classical case are considered in Lakshmikantham and Leela (1969), Szarski (1965), and Kamont (1980).

*Remark 2.* In Theorem 1 we can assume instead of (6) that

$$U_x(x, y) \leq f(x, y, U(x, y), U_y(x, y))$$
$$V_x(x, y) \geq f(x, y, V(x, y), V_y(x, y))$$

where for each $(x, y) \in \Gamma$ an equality may be attained in at most one place.

We introduce the following assumptions.

H5. The function $\sigma: [0, a) \times R_+ \to R_+$ is continuous and $\sigma(x, 0) = 0$ for $x \in [0, a)$.

H6. The right-hand maximum solution of the problem

$$W'(x) = \sigma(x, W(x)), \qquad W(0) = 0$$

is $W(x) = 0$, $x \in [0, a)$.

H7. $f$ satisfies the inequality

$$f(x, y, p, q) - f(x, y, \bar{p}, q) \geq -\sigma(x, \bar{p} - p) \qquad \text{on} \quad \Omega$$

where $p \leq \bar{p}$.

H8. There are functions $\sigma_i \in C([0, a) \times R_+, R_+)$, $i = 1, \ldots, k$, such that $\sigma_i(x, 0) = 0$ for $x \in [0, a)$ and

$$g_i(x, y, p) - g_i(x, y, \bar{p}) \geq -\sigma_i(x, \bar{p} - p) \qquad \text{on} \quad \tilde{\Omega}$$

where $p \leq \bar{p}$.

*Theorem 2.* Suppose that the following conditions hold:

1. Assumptions H1–H8 are met.
2. $U, V \in C^*_{\text{imp}}[E, R]$ satisfy the initial inequality

$$U(0, y) \leq V(0, y), \qquad \alpha(0) \leq y \leq \beta(0) \tag{14}$$

3. The differential inequalities

$$U_x(x, y) \leq f(x, y, U(x, y), U_y(x, y)) \\ V_x(x, y) \geq f(x, y, V(x, y), V_y(x, y)) \tag{15}$$

hold true on $\Gamma$ and

$$\begin{aligned} \Delta U(x_i, y) &\le g_i(x_i, y, U(x_i^-, y)) \\ \Delta V(x_i, y) &\ge g_i(x_i, y, V(x_i^-, y)) \end{aligned} \tag{16}$$

where $\alpha(x_i) \le y \le \beta(x_i)$, $i = 1, \ldots, k$.

Then we have

$$U(x, y) \le V(x, y) \qquad \text{on} \quad E$$

*Proof.* Suppose $a_0 \in (x_k, a)$. We prove that

$$U(x, y) \le V(x, y) \qquad \text{for} \quad (x, y) \in ([0, a_0) \times R^n) \cap E \tag{17}$$

Consider the problem

$$\begin{aligned} &W'(x) = \sigma(x, W(x)) + \varepsilon \\ &W(0) = \varepsilon \\ &W(x_i) = W(x_i^-) + \sigma_i(x_i, W(x_i^-)) + \varepsilon, \qquad i = 1, \ldots, k \end{aligned} \tag{18}$$

There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ there exists a solution $\omega(\cdot; \varepsilon)$ of (18) and this solution is defined on $[0, a_0)$.

Let $\tilde{V}(x, y) = V(x, y) + \omega(x; \varepsilon)$ for $(x, y) \in ([0, a_0) \times R^n) \cap E$. We prove that

$$U(x, y) < \tilde{V}(x, y) \qquad \text{on} \quad ([0, a_0) \times R^n) \cap E \tag{19}$$

We have that

$$\begin{aligned} \tilde{V}_x(x, y) &= V_x(x, y) + \omega'(x; \varepsilon) \\ &\ge f(x, y, V(x, y), V_y(x, y)) + \omega'(x; \varepsilon) \\ &= f(x, y, V(x, y), V_y(x, y)) + f(x, y, \tilde{V}(x, y), \tilde{V}_y(x, y)) \\ &\quad - f(x, y, \tilde{V}(x, y), \tilde{V}_y(x, y)) + \omega'(x; \varepsilon) \\ &> f(x, y, \tilde{V}(x, y), \tilde{V}_y(x, y)), \qquad (x, y) \in \Gamma \cap ([0, a_0) \times R^n) \end{aligned}$$

Thus we see that

$$\tilde{V}_x(x, y) > f(x, y, \tilde{V}(x, y), \tilde{V}_y(x, y)) \qquad \text{on} \quad \Gamma \cap ([0, a_0) \times R^n)$$

It follows that

$$\begin{aligned} &\tilde{V}(x_i, y) - \tilde{V}(x_i^-, y) \\ &\quad = V(x_i, y) + \omega(x_i; \varepsilon) - V(x_i^-, y) - \omega(x_i^-; \varepsilon) \\ &\quad \ge g_i(x_i, y, V(x_i^-, y)) + \omega(x_i; \varepsilon) - \omega(x_i^-; \varepsilon) \\ &\quad \ge g_i(x_i, y, \tilde{V}(x_i^-, y)) - \sigma_i(x_i, \omega(x_i^-; \varepsilon)) + \omega(x_i; \varepsilon) - \omega(x_i^-; \varepsilon) \\ &\quad > g_i(x_i, y, \tilde{V}(x_i^-, y)), \qquad \alpha(x_i) \le y \le \beta(x_i), \quad i = 1, \ldots, k \end{aligned}$$

Thus we have

$$\Delta \tilde{V}(x_i, y) > g_i(x_i, y, \tilde{V}(x_i^-, y))$$

where $\alpha(x_i) \leq y \leq \beta(x_i)$, $i = 1, \dots, k$.

Since $U(0, y) < \tilde{V}(0, y)$, $\alpha(0) \leq y \leq \beta(0)$, we deduce from Theorem 1 the assertion (19). Since $\lim_{\varepsilon \to 0} \omega(x; \varepsilon) = 0$ uniformly with respect to $x \in [0, a_0)$, we obtain (17). The constant $a_0 \in (x_k, a)$ is arbitrary; therefore the proof is complete. ■

We introduce the following assumptions:

H9. $\tilde{\sigma}: [0, a) \times R_- \to R_+$, $R_- = (-\infty, 0]$, is continuous and $\tilde{\sigma}(x, 0) = 0$ for $x \in [0, a)$.

H10. For $p \leq \bar{p}$ we have

$$f(x, y, p, q) - f(x, y, \bar{p}, q) \leq \tilde{\sigma}(x, p - \bar{p})$$

on $\Omega$.

H11. $\tilde{\sigma}_i: [0, a) \times R_- \to R_+$, $i = 1, \dots, k$, $\tilde{\sigma}_i$ are continuous and $\tilde{\sigma}_i(x, 0) = 0$ for $x \in [0, a)$.

H12. For $p \leq \bar{p}$ we have

$$g_i(x, y, p) - g_i(x, y, \bar{p}) \leq \tilde{\sigma}_i(x, p - \bar{p}) \qquad \text{on} \quad \tilde{\Omega}, \quad i = 1, \dots, k$$

H13. The left-hand minimum solution of the equation

$$W'(x) = \tilde{\sigma}(x, W(x))$$

satisfying the condition $\lim_{x \to a^-} W(x) = 0$ is $W(x) = 0$, $x \in [0, a)$.

*Theorem 3.* Suppose that the following conditions hold:

1. Assumptions H1–H4 and H9–H13 are met.
2. $U$, $V \in C^*_{\text{imp}}[E, R]$ satisfy the initial inequality (5) and the differential inequalities (15) hold on $\Gamma$.
3. Estimates (16) are satisfied.

Then we have

$$U(x, y) < V(x, y), \qquad (x, y) \in E \tag{20}$$

*Proof.* First we prove (20) for $(x, y) \in E \cap ([0, a - \varepsilon) \times R^n)$, where $a - x_k > \varepsilon > 0$. Let

$$0 < Z_0 < \min\{[V(0, y) - U(0, y)]: \alpha(0) \leq y \leq \beta(0)\}$$

For $\delta > 0$, denote by $\omega(\cdot\,; \delta)$ the right-hand minimum solution of the problem

$$\begin{aligned} &W'(x) = -\tilde{\sigma}(x, -W(x)) - \delta, \qquad W(0) = Z_0 \\ &W(x_i) = W(x_i^-) - \tilde{\sigma}_i(x_i, -W(x_i^-)) - \delta, \qquad i = 1, \ldots, k \end{aligned} \tag{21}$$

If $Z_0 > 0$ is fixed, then to every $\varepsilon > 0$ there corresponds $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ the solution $\omega(\cdot\,; \delta)$ of (21) exists and is positive on $[0, a - \varepsilon)$. Suppose that $\delta > 0$ is a constant such that $\omega(\cdot\,; \delta)$ satisfies the above conditions. Let

$$\tilde{U}(x, y) = U(x, y) + \omega(x; \delta)$$

for $(x, y) \in E \cap ([0, a - \varepsilon) \times R^n)$. We will prove that

$$\tilde{U}(x, y) < V(x, y) \qquad \text{on} \quad E \cap ([0, a - \varepsilon) \times R^n) \tag{22}$$

From assumptions H3 and H9–H13 it follows that

$$\begin{aligned} \tilde{U}_x(x, y) &= U_x(x, y) + \omega'(x; \delta) \\ &\le f(x, y, \tilde{U}(x, y), \tilde{U}_y(x, y)) + \omega'(x; \delta) \\ &\quad + [f(x, y, U(x, y), U_y(x, y)) - f(x, y, \tilde{U}(x, y), U_y(x, y))] \\ &\le f(x, y, \tilde{U}(x, y), \tilde{U}_y(x, y)) + \omega'(x; \delta) + \tilde{\sigma}(x, -\omega(x; \delta)) \\ &= f(x, y, \tilde{U}(x, y), \tilde{U}_y(x, y)) - \delta \end{aligned}$$

for $(x, y) \in \Gamma \cap ([0, a - \varepsilon) \times R^n)$. Then we have

$$\tilde{U}_x(x, y) < f(x, y, \tilde{U}(x, y), \tilde{U}_y(x, y))$$

for $(x, y) \in \Gamma \cap ([0, a - \varepsilon) \times R^n)$.

Now we prove that

$$\Delta \tilde{U}(x_i, y) < g_i(x_i, y, \tilde{U}(x_i^-, y)) \tag{23}$$

where $\alpha(x_i) \le y \le \beta(x_i)$; $i = 1, \ldots, k$.

From assumptions H9–H13 and from (16), (21) it follows that

$$\begin{aligned} \Delta \tilde{U}(x_i, y) &\le g_i(x_i, y, U(x_i^-, y)) + \omega(x_i; \delta) - \omega(x_i^-; \delta) \\ &\le g_i(x_i, y, \tilde{U}(x_i^-, y)) + \tilde{\sigma}_i(x_i, -\omega(x_i^-; \delta)) \\ &\quad + \omega(x_i; \delta) - \omega(x_i^-; \delta) \\ &= g_i(x_i, y, \tilde{U}(x_i^-, y)) - \delta \end{aligned}$$

where $\alpha(x_i) \le y \le \beta(x_i)$, $i = 1, \ldots, k$, which completes the proof of (23). Since $\tilde{U}(0, y) < V(0, y)$ for $\alpha(0) \le y \le \beta(0)$, then we have estimate (22) by virtue of Theorem 1.

Since $\varepsilon$ is arbitrary, inequality (20) holds true on $E$. ∎

### 3.2. Remarks on Strict Differential Inequalities

The papers of Haar (1928) and Nagumo (1938) initiated the theory of partial differential inequalities. The classical theory is described in detail in Lakshmikantham and Leela (1969), Ladde *et al.* (1985), and Szarski (1965). A theorem on strict differential inequalities is very important in this theory.

In Theorem 1 we assume that the impulses in differential inequality (6) are given on fixed hyperplanes $x = x_i$, $i = 1, \ldots, k$. Now we formulate a theorem on strict differential inequalities in a more general case.

Let $b = (b_1, \ldots, b_n)$ and $c = (c_1, \ldots, c_n)$, where

$$b_i \le \inf\{\alpha_i(x): x \in [0, a)\}, \qquad i = 1, \ldots, n$$

$$c_i \ge \sup\{\beta_i(x): x \in [0, a)\}, \qquad i = 1, \ldots, n$$

Suppose that we have $\psi = (\psi_1, \ldots, \psi_k) \in C([b, c], R^k)$ and

$$0 < \psi_1(y) < \psi_2(y) < \cdots < \psi_k(y) < a, \qquad y \in [b, c]$$

Let $\psi_0(y) = 0$ and $\psi_{k+1}(y) = a$ for $y \in [b, c]$.

We define

$$\tilde{\Gamma}_i = \{(x, y) \in E: \psi_i(y) < x < \psi_{i+1}(y)\}, \qquad i = 0, 1, \ldots, k$$

and

$$\tilde{\Gamma} = \tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 \cup \cdots \cup \tilde{\Gamma}_k$$

Let $C\psi[E, R]$ be the class of all functions $Z: E \to R$ such that:

(i) The functions $Z|_{\tilde{\Gamma}_i}$, $i = 0, 1, \ldots, k$, are continuous.

(ii) For each $i$, $1 \le i \le k$, $x = \psi_i(y)$ there exists

$$\lim_{\substack{(t,s) \to (x,y) \\ t < x}} Z(t, s) = Z(x^-, y), \qquad (x, y) \in E$$

(iii) For each $i$, $0 \le i \le k$, $x = \psi_i(y)$ there exists

$$\lim_{\substack{(t,s) \to (x,y) \\ t > x}} Z(t, s) = Z(x^+, y), \qquad (x, y) \in E$$

(iv) For each $i$, $0 \le i \le k$, $x = \psi_i(y)$, $(x, y) \in E$ we have

$$Z(x, y) = Z(x^+, y)$$

A function $Z \in C\psi[E, R]$ will be called a function of class $C^*\psi[E, R]$ if $Z$ has partial derivatives $Z_x(x, y)$, $Z_y(x, y)$ for $(x, y) \in \tilde{\Gamma}$ and there exists the total derivative of $Z$ on the set

$$S^* = \left\{ \bigcup_{i=0}^{k} [\partial E \cap \partial \tilde{\Gamma}_i] \right\} \cap [(0, a) \times R^n]$$

We introduce the following assumptions.

H14. $\alpha, \beta \in C([0, a), R^n)$ and $\alpha(x) < \beta(x)$ for $x \in [0, a)$.
H15. $\alpha, \beta \in C^1((0, a), R^n)$.

Now we formulate a generalization of Theorem 1.

*Theorem 4.* Suppose that the following conditions hold:

1. Assumptions H14, H15 are met.
2. $F: E \times R \times R^n \to R$; for $(x, y) \in S^*$, $p \in R$, we have

$$F(x, y, p, q) - F(x, y, p, \bar{q}) + \sum_{i \in I_-[x,y]} \alpha_i'(x)(q_i - \bar{q}_i) + \sum_{i \in I_+[x,y]} \beta_i'(x)(q_i - \bar{q}_i) \le 0$$

where $q = (q_1, \ldots, q_n)$, $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$, and

$$q_i \le \bar{q}_i \quad \text{for} \quad i \in I_-[x, y]$$
$$q_i \ge \bar{q}_i \quad \text{for} \quad i \in I_+[x, y]$$
$$q_i = \bar{q}_i \quad \text{for} \quad i \in I_0[x, y]$$

3. For each $i$, $1 \le i \le k$, the functions $g_i^*: E \times R \to R$ are such that $\delta_i^*(p) = p + g_i^*(x, y, p)$ are nondecreasing on $R$.
4. $U, V \in C^*\psi[E, R]$ satisfy the initial inequality (5), and the differential inequality

$$U_x(x, y) - F(x, y, U(x, y), U_y(x, y)) < V_x(x, y) - F(x, y, V(x, y), V_y(x, y))$$

holds true on $\tilde{\Gamma}$, and for each $i$, $1 \le i \le k$, $x = \psi_i(y)$, $(x, y) \in E$ we have

$$U(x_i, y) - U(x_i^-, y) - g_i^*(x_i, y, U(x_i^-, y)) < V(x_i, y) - V(x_i^-, y) - g_i^*(x_i, y, V(x_i^-, y))$$

Then we have

$$U(x, y) < V(x, y) \quad \text{for} \quad (x, y) \in E \tag{24}$$

We omit the proof of Theorem 4.

### 3.3. Comparison Theorems for Differential Inequalities

Here we prove theorems on estimates of functions satisfying impulsive partial differential inequalities by means of solutions of impulsive ordinary differential equations.

We define

$$P_i = (x_i, x_{i+1}), \quad i = 0, 1, \ldots, k; \qquad J = [0, a), \qquad P = P_0 \cup P_1 \cup \cdots \cup P_k$$

Let $C_{\text{imp}}[J, R]$ be the class of all functions $W: J \to R$ such that:

(i) The functions $W|_{P_i}$, $i = 0, 1, \ldots, k$, are continuous.
(ii) For each $i$, $1 \le i \le k$, $x = x_i$, there exists

$$\lim_{\substack{t \to x \\ t < x}} W(t) = W(x^-)$$

(iii) For each $i$, $0 \le i \le k$, $x = x_i$, there exists

$$\lim_{\substack{t \to x \\ t > x}} W(t) = W(x^+)$$

(iv) For each $i$, $0 \le i \le k$, we have $W(x_i) = W(x_i^+)$.

Let $S_x = \{y: (x, y) \in E\}$, $0 \le x < a$. For $Z \in C_{\text{imp}}[E, R]$ we define a function $TZ: [0, a) \to R_+$ by

$$(TZ)(x) = \max\{|Z(x, y)|: y \in S_x\}$$

For $q \in R^n$ we define $[q] = (|q_1|, \ldots, |q_n|)$.

*Lemma 1.* Suppose that the following conditions hold:

1. $\alpha, \beta \in C(J, R^n)$ and

$$\alpha(x) < \beta(x) \qquad \text{for } x \in J$$

2. $Z \in C_{\text{imp}}[E, R]$.

Then $TZ \in C_{\text{imp}}[J, R_+]$.

We omit the proof of Lemma 1.

We introduce the following assumptions:

H16. $\sigma: [0, a) \times R_+ \to R_+$ is a continuous function.
H17. $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k): [0, a) \times R_+ \to R_+^k$ is a continuous function.
H18. For each $i$, $i = 1, \ldots, k$, the function $\gamma_i(p) = p + \tilde{\sigma}_i(x, p)$ is nondecreasing on $R_+$.

*Lemma 2.* Suppose that the following conditions hold:

1. Assumptions H16–H18 are met.
2. $\varphi \in C_{\text{imp}}[J, R_+]$ and $\varphi(0) \le \eta_0$, $\eta_0 \in R_+$.
3. $\omega(\,\cdot\,; \eta_0): [0, a) \to R_+$ is the maximum solution of the problem

$$\begin{aligned} &W'(x) = \sigma(x, W(x)), \qquad x \in P \\ &W(0) = \eta_0 \\ &W(x_i) = W(x_i^-) + \tilde{\sigma}_i(x_i, W(x_i^-)), \qquad i = 1, \ldots, k \end{aligned} \tag{25}$$

4. For $x \in P$ we have

$$\mathscr{D}_- \varphi(x) \leq \sigma(x, \varphi(x)) \tag{26}$$

where $\mathscr{D}_-$ is the left-hand lower Dini derivative and

$$\varphi(x_i) \leq \varphi(x_i^-) + \tilde{\sigma}_i(x_i, \varphi(x_i^-)), \qquad i = 1, \ldots, k \tag{27}$$

Then we have

$$\varphi(x) \leq \omega(x; \eta_0), \qquad x \in [0, a)$$

We omit the simple proof of Lemma 2.
Introduce the following assumptions:

H19. $G: E \times R_+ \times R_+^n \to R_+$ is a continuous function.
H20. For $(x, y) \in E$ we have

$$G(x, y, p, q) - \sum_{i \in I_-[x,y]} \alpha_i'(x) q_i + \sum_{i \in I_+[x,y]} \beta_i'(x) q_i \leq \sigma(x, p) \tag{28}$$

where $q = (q_1, \ldots, q_n)$ and $q_i = 0$ for $i \in I_0[x, y]$.

*Remark 3.* If $E$ is the Haar pyramid

$$\{(x, y): x \in [0, a), |y_i| \leq a_i - L_i x, i = 1, \ldots, n\} \tag{29}$$

where $0 \leq aL_i < a_i$, $i = 1, \ldots, n$, and

$$G(x, y, p, q) = \sigma(x, p) + \sum_{i=1}^{n} L_i q_i$$

then condition (28) is satisfied.

*Theorem 5.* Suppose that the following conditions hold:

1. Assumptions H1–H2 and H16–H20 are met.
2. $U \in C^*_{\text{imp}}[E, R]$.
3. The differential inequality

$$|U_x(x, y)| \leq G(x, y, |U(x, y)|, [U_y(x, y)]) \tag{30}$$

for $(x, y) \in \Gamma$ and the initial estimate

$$|U(0, y)| \leq \eta_0, \qquad \eta_0 \in R_+ \tag{31}$$

where $y \in [\alpha(0), \beta(0)]$ are satisfied.
4. For each $i$, $1 \leq i \leq k$, and $\alpha(x_i) \leq y \leq \beta(x_i)$ we have

$$|U(x_i, y)| \leq |U(x_i^-, y)| + \tilde{\sigma}_i(x_i, |U(x_i^-, y)|) \tag{32}$$

5. The maximum solution $\omega(\cdot\,;\eta_0)$ of (25) exists on $[0, a)$. Then we have

$$|U(x, y)| \leq \omega(x; \eta_0), \qquad (x, y) \in E \tag{33}$$

*Proof.* Let us define $\varphi = TU$. Then $\varphi \in C_{\text{imp}}[J, R_+]$ and $\varphi(0) \leq \eta_0$. Now we prove that $\varphi$ satisfies (26). Suppose that $\tilde{x} \in P$. It follows that there exists $\tilde{y} \in [\alpha(\tilde{x}), \beta(\tilde{x})]$ such that

$$\varphi(\tilde{x}) = |U(\tilde{x}, \tilde{y})|$$

Suppose that $(\tilde{x}, \tilde{y}) \in \operatorname{Int} \Gamma$. Then $U_y(\tilde{x}, \tilde{y}) = 0$ and (28), (30) imply

$$\begin{aligned} \mathscr{D}_- \varphi(\tilde{x}) &\leq |U_x(\tilde{x}, \tilde{y})| \\ &\leq G(\tilde{x}, \tilde{y}, |U(\tilde{x}, \tilde{y})|, [U_y(\tilde{x}, \tilde{y})]) \leq \sigma(\tilde{x}, \varphi(\tilde{x})) \end{aligned}$$

Suppose that $(\tilde{x}, \tilde{y}) \in S$. There are two possibilities:

$$\varphi(\tilde{x}) = U(\tilde{x}, \tilde{y}) \tag{34}$$

$$\varphi(\tilde{x}) = -U(\tilde{x}, \tilde{y}) \tag{35}$$

Let us consider the case (34). We have

$$\begin{aligned} U_{y_i}(\tilde{x}, \tilde{y}) &\leq 0 \qquad \text{for} \quad i \in I_-[\tilde{x}, \tilde{y}] \\ U_{y_i}(\tilde{x}, \tilde{y}) &\geq 0 \qquad \text{for} \quad i \in I_+[\tilde{x}, \tilde{y}] \\ U_{y_i}(\tilde{x}, \tilde{y}) &= 0 \qquad \text{for} \quad i \in I_0[\tilde{x}, \tilde{y}] \end{aligned} \tag{36}$$

Let $\xi(x) = U(x, r(x))$, $x \in [0, \tilde{x}]$, where $r$ is given by (11). Then $\xi(x) \leq \varphi(x)$ for $x \in [0, \tilde{x}]$ and $\xi(\tilde{x}) = \varphi(\tilde{x})$. Therefore from (28), (30), and (36) we have that

$$\begin{aligned} \mathscr{D}_- \varphi(\tilde{x}) \leq \mathscr{D}_- \xi(\tilde{x}) &= U_x(\tilde{x}, \tilde{y}) + \sum_{i \in I_-[\tilde{x}, \tilde{y}]} \alpha_i'(\tilde{x}) U_{y_i}(\tilde{x}, \tilde{y}) \\ &\quad + \sum_{i \in I_+[\tilde{x}, \tilde{y}]} \beta_i'(\tilde{x}) U_{y_i}(\tilde{x}, \tilde{y}) \\ &\leq G(\tilde{x}, \tilde{y}, |U(\tilde{x}, \tilde{y})|, [U_y(\tilde{x}, \tilde{y})]) \\ &\quad + \sum_{i \in I_-[\tilde{x}, \tilde{y}]} \alpha_i'(\tilde{x}) U_{y_i}(\tilde{x}, \tilde{y}) + \sum_{i \in I_+[\tilde{x}, \tilde{y}]} \beta_i'(\tilde{x}) U_{y_i}(\tilde{x}, \tilde{y}) \\ &\leq \sigma(\tilde{x}, \varphi(\tilde{x})) \end{aligned}$$

Then we have (26). In a similar way we prove (26) in the case (35). Since $|U(x_i^-, y)| \leq \varphi(x_i^-)$ for $y \in [\alpha(x_i), \beta(x_i)]$, $i = 1, \ldots, k$, we have

$$\varphi(x_i) = |U(x_i, \tilde{y})| \leq \varphi(x_i^-) + \tilde{\sigma}_i(x_i, \varphi(x_i^-))$$

Consequently, the estimates (27) are satisfied. Now we obtain (33) from Lemma 2. ■

In the case when $E$ is the Haar pyramid (29) we have the following result.

*Theorem 6.* Suppose that the following conditions hold:

1. Assumptions H1, H2, and H16–H18 are met.
2. $U \in C^*_{\text{imp}}[E, R]$.
3. The differential inequality

$$|U_x(x, y)| \leq \sigma(x, |U(x, y)|) + \sum_{i=1}^{n} L_i |U_{y_i}(x, y)| \tag{37}$$

for $(x, y) \in \Gamma$ holds true.

4. The initial estimate (31) and conditions 4 and 5 of Theorem 5 are satisfied.

Then we have

$$|U(x, y)| \leq \omega(x; \eta_0)$$

for $(x, y) \in E$.

### 3.4. Estimates of Solutions of Differential Problems

In this section we give an application of Theorem 5.

*Theorem 7.* Suppose that the following conditions hold:

1. Assumptions H1–H2 and H16–H20 are met.
2. $G: E \times R_+ \times R^n_+ \to R_+$ and

$$|f(x, y, p, q)| \leq G(x, y, |p|, [q]) \qquad \text{on} \quad \Omega$$

3. $g = (g_1, \ldots, g_k): \tilde{\Omega} \to R^k$ are such that

$$[g(x, y, p)] \leq \tilde{\sigma}(x, |p|) \qquad \text{on} \quad \tilde{\Omega}$$

4. $\phi \in C([\alpha(0), \beta(0)], R)$, $U \in C^*_{\text{imp}}[E, R]$ is a solution of (1)–(3) and

$$|\phi(0, y)| \leq \eta_0, \qquad y \in [\alpha(0), \beta(0)], \qquad \eta_0 \in R_+$$

5. The maximum solution $\omega(\,\cdot\,; \eta_0)$ of (25) is defined on $[0, a)$.

Then we have

$$|U(x, y)| \leq \omega(x; \eta_0) \qquad \text{on} \quad E$$

*Proof.* The solution $U$ of (1)–(3) satisfies all assumptions of Theorem 5 and the statement follows. ■

### 3.5. Uniqueness Criteria

Let us consider two problems—problem (1)–(3) and the problem

$$Z_x(x, y) = \tilde{f}(x, y, Z(x, y), Z_y(x, y)) \tag{38}$$

$$Z(0, y) = \tilde{\phi}(y), \qquad y \in [\alpha(0), \beta(0)] \tag{39}$$

$$\Delta Z(x_i, y) = \tilde{g}_i(x_i, y, Z(x_i^-, y)), \qquad \alpha(x_i) \le y \le \beta(x_i), \quad i = 1, \dots, k \tag{40}$$

where $\tilde{f}: \Omega \to R$, $\tilde{\phi}: [\alpha(0), \beta(0)] \to R$, and $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_k): \tilde{\Omega} \to R^k$ are given functions.

*Theorem 8.* Suppose that the following conditions hold:

1. Assumptions H1, H2, and H16–H20 are met.
2. The functions $f, \tilde{f}, g, \tilde{g}$ are such that

$$|f(x, y, p, q) - \tilde{f}(x, y, \bar{p}, \bar{q})| \le G(x, y, |p - \bar{p}|, [q - \bar{q}])$$

on $\Omega$, and

$$[g(x, y, p) - \tilde{g}(x, y, \bar{p})] \le \tilde{\sigma}(x, |p - \bar{p}|)$$

on $\tilde{\Omega}$.

3. $\eta_0 \in R_+$ and

$$|\phi(0, y) - \tilde{\phi}(0, y)| \le \eta_0 \qquad \text{for} \quad y \in [\alpha(0), \beta(0)]$$

4. The maximum solution $\omega(\,\cdot\,; \eta_0)$ of (25) is defined on $[0, a)$.
5. $U, \tilde{U} \in C^*_{\mathrm{imp}}[E, R]$ are solutions of (1)–(3) and (38)–(40) respectively.

Then we have

$$|U(x, y) - \tilde{U}(x, y)| \le \omega(x; \eta_0) \qquad \text{on} \quad E$$

*Proof.* If we put $\tilde{Z} = U - \tilde{U}$, then $\tilde{Z}$ satisfies all the conditions of Theorem 5 and the statement follows. ■

*Theorem 9.* Suppose that the following conditions hold:

1. Assumptions H1, H2, and H16–H20 are met.
2. $f: \Omega \to R$ and $g: \tilde{\Omega} \to R^k$ are such that

$$|f(x, y, p, q) - f(x, y, \bar{p}, \bar{q})| \le G(x, y, |p - \bar{p}|, [q - \bar{q}]) \qquad \text{on} \quad \Omega \tag{41}$$

$$[g(x, y, p) - g(x, y, \bar{p})] \le \tilde{\sigma}(x, |p - \bar{p}|) \qquad \text{on} \quad \tilde{\Omega} \tag{42}$$

3. $\sigma(x, 0) = 0$ and $\tilde{\sigma}(x, 0) = 0$ for $x \in [0, a)$ and the maximum solution of the problem

$$W'(x) = \sigma(x, W(x)), \qquad W(0) = 0$$
$$W(x_i) = W(x_i^-) + \tilde{\sigma}_i(x_i, W(x_i^-)), \qquad i = 1, \ldots, k$$

is $W(x) = 0$, $x \in J$.

Then the Cauchy problem (1)–(3) admits at most one solution $U$ which is of class $C^*_{\text{imp}}[E, R]$.

*Proof.* If we put $\tilde{f} = f$ and $\tilde{g} = g$, then we deduce our theorem from Theorem 8. ■

*Remark 4.* Suppose $E$ is given by (29). If assumptions H1, H2, and H16–H18 are met and

$$G(x, y, p, q) = \sigma(x, p) + \sum_{i=1}^{n} L_i q_i \qquad \text{on} \quad \Omega$$

then condition (41) has the form

$$|f(x, y, p, q) - f(x, y, \bar{p}, \bar{q})|$$
$$\leq \sigma(x, |p - \bar{p}|) + \sum_{i=1}^{n} L_i |q_i - \bar{q}_i| \qquad \text{on} \quad \Omega$$

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